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Integral Solution of Sextic Equation With Five Unknowns

$$x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$$

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Abstract

We obtain infinitely many many non-zero integer quintuples (x, y, z, w, T) satisfying the non-homogeneous sextic equation with five unknowns $x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$.Various interesting properties among the values of x, y, z, w and T are presented.

Keywords: Sextic equation with five unknowns, integral solutions.

Introduction

The theory of diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly, in [5, 6], sextic equations with 3 unknowns are studied for their integral solutions. [7, 9] analyse sextic equations with 4 unknowns for their non-zero integer solutions. This communication analyses a sextic equation with 5 unknowns given by $x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$. Infinitely many non-zero integer quintuples (x, y, z, w, T) satisfying the above equation are obtained. Various interesting properties among the values of x, y, z, w and T are presented.

Method of Analysis

The sextic equation with five unknowns to be solved is

$$x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5 \quad (1)$$

The introduction of the linear transformations,

$$x = u+v, y = u-v, z = u+p, w = u-p, u \neq v \neq p \quad (2)$$

in (1) leads to

$$v^2 - p^2 = T^5 \quad (3)$$

It is observed that (3) has infinitely many integral solutions. For simplicity and clear understanding, we exhibit below the different choices for v and p satisfying (3), when

(i)T is an odd integer

(ii)T is an even integer and

(iii)T is a perfect square.

Knowing the values of v and p and using (2), we get infinitely many non-zero distinct integral solutions of (1)

Case (i): Let $T = 2k+1 \quad (4)$

Employing the identity

$$(A+1)^2 - A^2 = 2A+1$$

and performing a simple algebra, the values of v and p are given by,

$$v = 16k^5 + 40k^4 + 40k^3 + 20k^2 + 5k + 1$$

$$p = 16k^5 + 40k^4 + 40k^3 + 20k^2 + 5k$$

Substituting the values of v and p in (2), one obtains the non-zero integral solutions of (1). However on applying the method of factorisation, two more choices for v and p satisfying (3) are obtained and they are as follows:

Choice1:

$$v = 8k^4 + 16k^3 + 12k^2 + 5k + 1$$
$$v = 8k^4 + 16k^3 + 12k^2 + 3k$$

Choice2:

$$v = 4k^3 + 8k^2 + 5k + 1$$
$$p = 4k^3 + 4k^2 + k$$

As mentioned above, one can obtain the corresponding integral solutions of (1).

Case (ii): Let $T = 2k$ (5)

For simplicity and brevity, we present below different choices for v and p satisfying (3).

(a).First of all, it is observed that the application of the identity

$$(A+2^i)^2 - A^2 = 2^{i+1}(A+2^{i-1}), i = 1,2,3,4.$$

yields the following four sets of values for v and p:

v	p
$8k^5+1$	$8k^5-1$
$4k^5+2$	$4k^5-2$
$2k^5+4$	$2k^5-4$
k^5+8	k^5-8

(b) Secondly, on applying the method of factorisation, two more choices of v and p satisfying (3) are obtained and they are as follows:

Choice1:

$$v = 8k^4 + k$$
$$p = 8k^4 - k$$

Choice2:

$$v = 4k^3 + 2k^2$$
$$p = 4k^3 - 2k^2$$

(c). Also, introducing the linear transformations

$$v = 2k + s, \; p = 2k - s$$

in (3) and performing a simple algebra, we have,

$$v = 2k + 4k^4$$
$$p = 2k - 4k^4$$

Substituting each of the above values of v and p in (2), we obtain the corresponding solutions of (1).

Case (iii): Take $T = \alpha^2$ (6)

Substituting (6) in (3) we get,

$$v^2 = p^2 + (\alpha^5)^2 \quad (7)$$

which is in the form of Pythagorean equation and is satisfied by

$$\alpha^5 = 2rs, \; p = r^2 - s^2, v = r^2 + s^2, r > s > 0 \quad (8)$$

Choose r and s such that $rs = 16\beta^5$ (9)

and thus $\alpha = 2\beta$.

Knowing the values of r, s and using (8) and (2), the corresponding solutions of (1) are obtained. For illustration, take

$$r = 2^3\beta^4, s = 2\beta, r > s > 0$$

The corresponding solutions are given by

$$x = u + 64\beta^8 + 4\beta^2$$
$$y = u - 64\beta^8 - 4\beta^2$$
$$z = u + 64\beta^8 - 4\beta^2$$
$$w = u - 64\beta^8 + 4\beta^2$$
$$T = 4\beta^2$$

It is to be noted that, the solutions of (7) may also be written as

$$v = r^2 + s^2, \; p = 2rs, \alpha^5 = r^2 - s^2, r > s > 0$$

Choose $r = \dfrac{\alpha^3 + \alpha^2}{2}, s = \dfrac{\alpha^3 - \alpha^2}{2}$

Substituting the values of r, s in (10) and using (2), the corresponding integral solutions of (1) are obtained. It is to be noted that, in addition to the above choices for v and p, we have two more choices to obtain v and p, which are illustrated below:

Illustration1:

The assumption $v = TV, \; p = TP$ in (3) yields

$$v = m(m^2 - n^2), \; p = n(m^2 - n^2), T = m^2 - n^2$$

From (11), (12) and (2), the corresponding integral solutions of (1) are

$$x = u + m(m^2 - n^2)^2$$
$$y = u - m(m^2 - n^2)^2$$
$$z = u + n(m^2 - n^2)^2$$
$$w = u - n(m^2 - n^2)^2$$
$$T = m^2 - n^2$$

Illustration2:

The assumption $v = T^2V, \; p = T^2P$ in (3) yields

$$v = 2mn(m^2 - n^2)^4, \; p = (m^2 + n^2)(m^2 - n^2)^4, T = -(m^2 - n^2)^2$$

From (13), (14) and (2), the corresponding integral solutions are

$x = u + 2mn(m^2 - n^2)^4$

$y = u - 2mn(m^2 - n^2)^4$

$z = u + (m^2 + n^2)(m^2 - n^2)^4$

$w = u - (m^2 + n^2)(m^2 - n^2)^4$

$T = -(m^2 - n^2)^2$

Properties

Each of the above solutions pattern satisfies the following relations:

1. $(x-y)^2 - (z-w)^2 = 4T^5$
2. $x^2 - y^2 - (y+w)(x-z) = 2T^5$
3. $24(x+w)(y+z) + 6(x-y+w+z)^2$ is a nasty number.
4. $16(x^2 + y^2 - z^2 - w^2)$ is a quintic number.
5. R_1 : Rectangle of dimensions x, y with area A_1 and perimeter P_1.

   R_2 : Rectangle of dimensions z, w with area A_2 and perimeter P_2.

   (i).Then $A_1 - A_2$ is a quintic integer.

   (ii). $P_1 = P_2$.
6. $xy(x^2 - y^2)$ = 4(area of the Pythagorean triangle with generators u, v).
7. $zw(z^2 - w^2)$ = 4(area of the Pythagorean triangle with generators u, p).

Conclusion

In conclusion, one may get different patterns of solutions to (1) and their corresponding properties.

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